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q-EXTENSION OF THE p-ADIC GAMMA FUNCTION

BY

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ABSTRACT. p-adic functions depending on a parameter q, $0 < |q-1|_p < 1$, are defined which extend Y. Morita's p-adic gamma function and the derivative of J. Diamond's p-adic log-gamma function in the same way as the classical q-gamma function $\Gamma_q(x)$ extends $\Gamma(x)$. Properties of these functions which are analogous to the basic identities satisfied by $\Gamma_q(x)$ are developed.

1. Introduction. A generalized gamma function $\Gamma_q(x)$, depending on a parameter $0 < q < 1$, was introduced and studied by F. H. Jackson, R. Askey, G. E. Andrews and others. Defined as

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1-q^x)(1-q^{x+1})(1-q^{x+2})\cdots},$$

it satisfies relations which generalize the well-known identities for the gamma function, and in the limit as $q \to 1^-$ it becomes $\Gamma(x)$.

The purpose of this article is to construct and study the properties of a natural q-extension $\Gamma_{p,q}$ of Morita's p-adic gamma function Γ_p [13] and a q-extension $\psi_{p,q}$ of the derivative of Diamond's p-adic log-gamma function G_p [4]. Recall that Γ_p is a function from the p-adic integers $\mathbf{Z}_p$ to the p-adic units $\mathbf{Z}_p^*$ defined by

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{j<n}' j,$$

where n runs over positive integers and $\prod'$ means that indices j divisible by p are omitted. On the other hand, G_p is a function on the *complement* of $\mathbf{Z}_p$ in Ω_p (where Ω_p is the p-adic completion of the algebraic closure of the field $\mathbf{Q}_p$ of p-adic numbers, with norm $|\ |_p$ for which $|p|_p = p^{-1}$). It is defined as

$$G_p(x) = \lim_{N \to \infty} p^{-N} \sum_{0 \leq j < p^N} (x+j)(\log_p(x+j)-1),$$

where $\log_p$ is the Iwasawa p-adic logarithm [7]. Although G_p is *not* $\log_p \Gamma_p$, it has the following two connections with $\log_p \Gamma_p$ (see [4] and [6]). (1) If we let $G_p^*(x) = G_p(x) - G_p(x/p)$, which is defined on the complement of $\mathbf{Z}_p^*$ by the same formula as G_p with Σ replaced by Σ', then $G_p^*(x) = \log_p \Gamma_p(x)$ for $x \in p\mathbf{Z}_p$. (2) $\log_p \Gamma_p(x) = \sum_{0 \leq j < p,\, x+j \notin p\mathbf{Z}_p} G_p((x+j)/p)$ for $x \in \mathbf{Z}_p$.

For simplicity suppose $p > 2$.

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2. q-extension of Γ_p.

THEOREM 1. *Let $t \in \Omega_p$, $|t|_p < 1$, $t \neq 0$. Set $q = 1 + t$, and for any n define*

$$\Gamma_{p,q}(n) = (-1)^n \prod_{j<n}{}' \frac{1-q^j}{1-q}.$$

Then $\Gamma_{p,q}$ extends to a continuous function on $\mathbf{Z}_p$, and $\lim_{q\to 1} \Gamma_{p,q} = \Gamma_p$.

PROOF. We first prove two general lemmas.

LEMMA 1. *Let $P_1(X), \ldots, P_h(X) \in \mathbf{Q}[X]$. Then there exists $Q(X) \in \mathbf{Q}[X]$ such that for all n*

$$\sum_{i_1<i_2<\cdots<i_h<n} P_1(i_1) \cdots P_h(i_h) = Q(n).$$

PROOF OF LEMMA. Use induction on h. For $h = 1$ the lemma follows because $\sum_{i_1=1}^{n-1} i_1^l$ is a polynomial in n for any l. Suppose the lemma holds for $h - 1$. We have

$$\sum_{i_1<i_2<\cdots<i_h<n} P_1(i_1) \cdots P_h(i_h) = \sum_{i_2<\cdots<i_h<n} \left(\sum_{i_1<i_2} P_1(i_1) \right) P_2(i_2) \cdots P_h(i_h)$$

$$= \sum_{i_2<\cdots<i_h<n} (Q_1(i_2)P_2(i_2))P_3(i_3) \cdots P_h(i_h)$$

by the lemma for $h = 1$. But this is of the form $Q(n)$ by the lemma for $h - 1$ applied to the polynomials $Q_1P_2, P_3, \ldots, P_h$.

LEMMA 2. *Let $P_k(X) \in \mathbf{Q}[X]$, $k = 1, 2, \ldots$; $A_j(t) = 1 + \sum_{k=1}^{\infty} P_k(j)t^k \in \mathbf{Q}[[t]]$, $j = 1, 2, \ldots$. Then there exist $Q_k(X) \in \mathbf{Q}[X]$ such that for all n,*

$$\prod_{j<n} A_j(t) = 1 + \sum_{k=1}^{\infty} Q_k(n)t^k.$$

PROOF OF LEMMA. For each k let $s = \{s_1, \ldots, s_h\}$ run through the set S of partitions of ordered positive integers s_i whose sum is k. The coefficient of t^k in $\prod_{j<n} A_j(t)$ is clearly

$$\sum_{s\in S} \sum_{i_1<i_2<\cdots<i_h<n} P_{s_1}(i_1) \cdots P_{s_h}(i_h).$$

Since the first sum is finite, this is a polynomial in n by Lemma 1.

PROOF OF THEOREM 1. Let $P_k(X) = (X-1)(X-2)\cdots(X-k)/(k+1)!$. Then

$$\frac{1-q^j}{1-q} = \frac{(1+t)^j - 1}{t} = j\left(1 + \sum_{k=1}^{\infty} P_k(j)t^k\right).$$

Further let $\tilde{n} = [(n-1)/p] + 1$, and let $\tilde{P}_k(X) = P_k(pX)$. Finally, let

$$A_j(t) = 1 + \sum_{k=1}^{\infty} P_k(j)t^k, \qquad \tilde{A}_j(t) = 1 + \sum_{k=1}^{\infty} \tilde{P}_k(j)t^k.$$

Then

$$\Gamma_{p,q}(n) = \Gamma_p(n)\frac{\prod_{j<n} A_j(t)}{\prod_{j<\tilde{n}} \tilde{A}_j(t)}.$$

By Lemma 2, the quotient can be written $(1 + \sum Q_k(n)t^k)/(1 + \sum \tilde{Q}_k(\tilde{n})t^k)$.

Note that $\Gamma_{p,q}(n) = (-1)^n \prod_{j<n, p\nmid j}(1 + (1 + t) + \cdots + (1 + t)^{j-1}) \in \mathbf{Z}[t]$, and hence $\Gamma_{p,q}(n)/\Gamma_p(n) = 1 + \sum_{k=1}^{\infty} R_k(n)t^k \in \mathbf{Z}_p[[t]]$ for each n (where the R_k are some functions of n), i.e., $|R_k(n)|_p \leq 1$ for all k, n. Since each R_k is a finite expression in Q_{k_1} and $\tilde{Q}_{k_2}$, $k_1, k_2 \leq k$, and since each $\tilde{Q}_{k_2}$ depends continuously on $\tilde{n}$ and hence on n, and Q_{k_1} is just a polynomial in n, it follows that R_k is a bounded (by 1) continuous function of n. Since $|t|_p < 1$, the theorem follows. Q.E.D.

THEOREM 2.

(1)

$$\Gamma_{p,q}(x + 1)/\Gamma_{p,q}(x) = \begin{cases} -(1 - q^x)/(1 - q) & \text{if } x \in \mathbf{Z}_p^*, \\ -1 & \text{if } x \in p\mathbf{Z}_p. \end{cases}$$

(2)

$$\Gamma_{p,q}(x)\Gamma_{p,q^m}\left(\frac{1}{m}\right)\Gamma_{p,q^m}\left(\frac{2}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{m-1}{m}\right)$$
$$= \Gamma_{p,q^m}\left(\frac{x}{m}\right)\Gamma_{p,q^m}\left(\frac{x+1}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{x+m-1}{m}\right)\left(\frac{1-q^m}{1-q}\right)^{x-\tilde{x}}$$

for any $x \in \mathbf{Z}_p$ and any positive integer m prime to p, where $\tilde{}$ is the unique continuous extension to $\mathbf{Z}_p$ of the map $n \mapsto \tilde{n} = [(n - 1)/p] + 1$ used in the proof of Theorem 1. (If $x = a_0 + a_1 p + a_2 p^2 + \cdots$, then $\tilde{x} = (a_1 + 1) + a_2 p + a_3 p^2 + \cdots$ if $a_0 \neq 0$ and $a_1 + a_2 p + a_3 p^2 + \cdots$ if $a_0 = 0$. Note that $(1 - q^m)/(1 - q)$ is a p-adic unit, and the exponent can be written $a_0 - p + (p - 1)\tilde{x}$ if $a_0 \neq 0$ and $(p - 1)\tilde{x}$ if $a_0 = 0$.)

(3)

$$\Gamma_{p,q}(x)\Gamma_{p,1/q}(1 - x) = -(-q)^{\tilde{x}-x}.$$

PROOF. Since both sides of (1), (2) and (3) are continuous in x, it suffices to prove them for $x = n$. In that case (1) follows immediately from the definition.

(2) For fixed m let A_n denote the left side of (3) for $x = n$, and let B_n denote the right side. We prove that $A_n = B_n$ by induction on n. Trivially $A_1 = B_1$. Suppose $A_n = B_n$. We have

$$\frac{A_{n+1}}{A_n} = \frac{\Gamma_{p,q}(n + 1)}{\Gamma_{p,q}(n)} = \begin{cases} -(1 - q^n)/(1 - q) & \text{if } p\nmid n, \\ -1 & \text{if } p|n, \end{cases}$$

and

$$\frac{B_{n+1}}{B_n} = \frac{\Gamma_{p,q^m}(n/m + 1)}{\Gamma_{p,q^m}(n/m)}\left(\frac{1-q^m}{1-q}\right)^{1+\tilde{n}-\widetilde{(n+1)}}$$
$$= \begin{cases} (-(1 - q^n)/(1 - q^m)) \cdot ((1 - q^m)/(1 - q)) & \text{if } p\nmid n, \\ -1 \cdot 1 & \text{if } p|n. \end{cases}$$

Hence $A_{n+1}/A_n = B_{n+1}/B_n$, and so $A_{n+1} = B_{n+1}$. This completes the induction.

(3) is easily proved by induction in the same way as (2). Q.E.D.

REMARKS. 1. Comparing parts (2) and (3) of Theorem 2 gives support for a comment once made by B. H. Gross that the Euler reflection formula for the gamma function should be thought of as the (−1)-case of the multiplication formula. The right side of (3) can be written as

$$\Gamma_{p,q}(1)\left(\frac{1-q^{-1}}{1-q}\right)^{x-\tilde{x}}$$

(recall $\Gamma_{p,q}(1) = -1$). This similarity between the multiplication and the reflection formulas for Γ_p was not clear until we looked at its q-extension $\Gamma_{p,q}$.

2. In the classical case the type of argument in Theorem 2 above will quickly reveal that $\Gamma_q(x)\Gamma_{1/q}(1-x)(-q)^x$ is periodic of period 1 (where we use D. Moak's Γ_q for $q > 1$ [**11**] to define $\Gamma_{1/q}(1-x)$). But this periodic function is not a constant (as any continuous function on $\mathbf{Z}_p$ with period 1 must be); it turns out to equal a constant divided by the theta-function

$$\sum_{n=-\infty}^{\infty} (-1)^{-(n+x)} q^{(n+x-1/2)^2/2}.$$

3. Table of properties.

Classical Case

gamma function	its q-extension $(0 < q < 1)$
(1) $\Gamma(n+1) = n!$	$\Gamma_q(n+1) = \prod_{j=1}^{n} (1 + q + q^2 + \cdots + q^{j-1})$
(2) $\dfrac{\Gamma(x+1)}{\Gamma(x)} = x, \quad \Gamma(1) = 1$	$\dfrac{\Gamma_q(x+1)}{\Gamma_q(x)} = \dfrac{(1-q^x)}{(1-q)}, \quad \Gamma(1) = 1.$

Γ and Γ_q are uniquely characterized by (2) together with convexity of their logarithm.

(3) $\Gamma(x)\Gamma\left(\frac{1}{m}\right)\cdots\Gamma\left(\frac{m-1}{m}\right) = \Gamma\left(\frac{x}{m}\right)\Gamma\left(\frac{x+1}{m}\right)\cdots\Gamma\left(\frac{x+m-1}{m}\right)m^{x-1}$ for any positive integer m	$\Gamma_q(x)\Gamma_{q^m}\left(\frac{1}{m}\right)\cdots\Gamma_{q^m}\left(\frac{m-1}{m}\right) = \Gamma_{q^m}\left(\frac{x}{m}\right)\Gamma_{q^m}\left(\frac{x+1}{m}\right)\cdots\Gamma_{q^m}\left(\frac{x+m-1}{m}\right)\left(\frac{(1-q^m)}{(1-q)}\right)^{x-1}$ for any positive integer m
(4) $\Gamma(x)\cdot\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$	$\Gamma_q(x)\Gamma_q(1-x) = \sum_{n=-\infty}^{\infty} \dfrac{(1-q)(-1)^n q^{n(n+1)/2}}{(1-q^{n+x})}$
(5) $\operatorname{dlog}\Gamma(1) = -\gamma$	$\operatorname{dlog}\Gamma_q(1) = -\gamma_q.$

p-adic Case

gamma function	its q-extension $(0 < \lvert q-1\rvert_p < 1)$
(1) $\Gamma_p(n+1) = (-1)^{n+1}\prod_{j=1}^{n}{}' j$	$\Gamma_{p,q}(n+1) = (-1)^{n+1}\prod_{j=1}^{n}{}' \cdot (1+q+q^2+\cdots+q^{j-1})$
(2) $\dfrac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x \text{ if } x\in \mathbf{Z}_p^*, \\ -1 \text{ if } x \in p\mathbf{Z}_p, \end{cases}$ $\Gamma_p(1) = -1$	$\dfrac{\Gamma_{p,q}(x+1)}{\Gamma_{p,q}(x)} = \begin{cases} -\dfrac{(1-q^x)}{(1-q)} \text{ if } x\in \mathbf{Z}_p^*, \\ -1 \text{ if } x \in p\mathbf{Z}_p, \end{cases}$ $\Gamma_{p,q}(1) = -1.$

Γ_p and $\Gamma_{p,q}$ are uniquely characterized by (2) together with continuity.

(3) $\Gamma_p(x)\Gamma_p\left(\frac{1}{m}\right)\cdots\Gamma_p\left(\frac{m-1}{m}\right) = \Gamma_p\left(\frac{x}{m}\right)\Gamma_p\left(\frac{x+1}{m}\right)\cdots \Gamma_p\left(\frac{x+m-1}{m}\right)m^{x-\bar{x}}$ for any positive integer m prime to p	$\Gamma_{p,q}(x)\Gamma_{p,q^m}\left(\frac{1}{m}\right)\cdots\Gamma_{p,q^m}\left(\frac{m-1}{m}\right) = \Gamma_{p,q^m}\left(\frac{x}{m}\right)\Gamma_{p,q^m}\left(\frac{x+1}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{x+m-1}{m}\right)\left(\frac{1-q^m}{1-q}\right)^{x-\bar{x}}$ for any positive integer m prime to p
(4) $\Gamma_p(x)\cdot\Gamma_p(1-x) = (-1)^{x-\bar{x}}\Gamma_p(1)$	$\Gamma_{p,q}(x)\cdot\Gamma_{p,1/q}(1-x) = (-q)^{\bar{x}-x}\Gamma_{p,q}(1)$
(5) $\operatorname{dlog}\Gamma_p(0) = -(1-1/p)\gamma_p$	$\operatorname{dlog}\Gamma_{p,q}(0) = -(1-1/p)\gamma_{p,q}$ (see below)

Finally, $\Gamma_q \to \Gamma$ as $q \to 1^-$, and $\Gamma_{p,q} \to \Gamma_p$ as $q \to 1$.

4. q-extension of G_p'. For $x \in \Omega_p$ let $d(x) = \min_{z\in \mathbf{Z}_p}\lvert x-z\rvert_p$ be the distance from x to $\mathbf{Z}_p$. For $q = 1+t$, $\lvert t\rvert_p < 1$, let $\varepsilon_q = 1/(\lvert\log_p q\rvert_p \cdot p^{1/(p-1)})$, and let $D_q = \{x \in \Omega_p \mid 0 < d(x) < \varepsilon_q\}$. Thus, if $\operatorname{ord}_p t > 1/(p-1)$, we have $\varepsilon_q = 1/(\lvert t\rvert_p\cdot p^{1/(p-1)}) > 1$ and $D_q = \{x \in \Omega_p - \mathbf{Z}_p \mid \lvert x\rvert_p < \varepsilon_q\}$; as $q \to 1$, D_q expands to all of $\Omega_p - \mathbf{Z}_p$.

If $x \in D_q$, say $\lvert x - z\rvert_p < \varepsilon_q$ with $z \in \mathbf{Z}_p$, then $q^x = q^z e^{(x-z)\log_p q}$, and so q^x is a locally analytic function on D_q. Since also $q^x \neq 1$ here, it follows that $\log_p(1-q^x)$ is locally analytic on D_q. We can then define

$$\psi_{p,q}(x) = \lim_{N\to\infty} p^{-N} \sum_{0<j<p^N} \log_p \frac{1-q^{x+j}}{1-q}$$
$$= -\log_p(1-q) + \lim_{N\to\infty} p^{-N}\sum_{0<j<p^N}\log_p(1-q^{x+j}),$$

which exists and is locally analytic on D_q by Theorem 2 of [4]. Clearly $\psi_{p,q} \to G_p'$ as $q \to 1$.

We can define $\psi_{p,q}^*(x)$–a q-extension of $G_p^{*\prime}$–by replacing $d(x)$ by $d^*(x) = \min_{z \in \mathbf{Z}_p^*} |x - z|_p$, D_q by $D_q^* = \{x \in \Omega_p | 0 < d^*(x) < \varepsilon_q\}$, and Σ by Σ' in the definition of $\psi_{p,q}$. Then

$$\psi_{p,q}^*(x) = \psi_{p,q}(x) - \frac{1}{p}\psi_{p,q^p}\left(\frac{x}{p}\right) - \frac{1}{p}\log_p \frac{1-q^p}{1-q} \qquad \text{for } x \in D_q.$$

We first show the relationship between $\psi_{p,q}(x)$ and $d \log_p \Gamma_{p,q}(x)/dx$.

THEOREM 3.

(1)

$$\psi_{p,q}^*(x) = \frac{d}{dx}\log_p \Gamma_{p,q}(x) \quad \textit{for } x \in p\mathbf{Z}_p;$$

(2)

$$\frac{d}{dx}\log_p \Gamma_{p,q}(x) = \frac{1}{p}\sum_{\substack{0<j<p \\ x+j \notin p\mathbf{Z}_p}} \psi_{p,q^p}\left(\frac{x+j}{p}\right) + \left(1 - \frac{1}{p}\right)\log_p \frac{1-q^p}{1-q} \quad \textit{for } x \in \mathbf{Z}_p.$$

Note that if $\operatorname{ord}_p(q-1) > 1/(p-1)$, then $\varepsilon_q > 1$ and $\varepsilon_{q^p} > p$, so that $x \in D_q^*$ and $(x+j)/p \in D_{q^p}$. But even if $\operatorname{ord}_p(q-1) \leq 1/(p-1)$, in which case we may have $x \notin D_q^*$ and $(x+j)/p \notin D_{q^p}$, still $\psi_{p,q}^*(x)$ in (1) and $\psi_{p,q^p}((x+j)/p)$ in (2) are well defined because $x \in p\mathbf{Z}_p$ and $x \in \mathbf{Z}_p$, respectively.

PROOF. (1) By the definition of $\Gamma_{p,q}$, for $x \in p\mathbf{Z}_p$ we have

$$p^{-N}\left(\log_p \Gamma_{p,q}(x+p^N) - \log_p \Gamma_{p,q}(x)\right) = p^{-N} \sum_{0<j<p^N}{}' \log_p \frac{1-q^{x+j}}{1-q},$$

and taking the limit as $N \to \infty$ gives (1).

(2) The sum on the right in (2) is equal to

$$\lim_{N\to\infty} p^{-N-1} \sum_{\substack{0<j<p^N \\ 0<k<p \\ x+k \notin p\mathbf{Z}_p}} \log_p \frac{1-q^{p((x+k)/p+j)}}{1-q^p}$$

$$= -\left(1-\frac{1}{p}\right)\log_p \frac{1-q^p}{1-q} + \lim_{N\to\infty} p^{-N} \sum_{\substack{0<j<p^N \\ x+j \notin p\mathbf{Z}_p}} \log_p \frac{1-q^{x+j}}{1-q}$$

$$= -\left(1-\frac{1}{p}\right)\log_p \frac{1-q^p}{1-q} + \lim_{N\to\infty} p^{-N}\left(\log_p \Gamma_{p,q}(x+p^N) - \log_p \Gamma_{p,q}(x)\right)$$

$$= -\left(1-\frac{1}{p}\right)\log_p \frac{1-q^p}{1-q} + \frac{d}{dx}\log_p \Gamma_{p,q}(x). \quad \text{Q.E.D.}$$

The next theorem, giving identities for $\psi_{p,q}$, should be compared with the table in §3.

THEOREM 4. *For* $x \in D_q$,
(1)

$$\psi_{p,q}(x+1) - \psi_{p,q}(x) = -\frac{q^x \log_p q}{1-q^x} = \frac{d}{dx} \log_p \frac{1-q^x}{1-q},$$

(2)

$$\psi_{p,q}(x) - \frac{1}{m}\sum_{h=0}^{m-1} \psi_{p,q^m}\left(\frac{x+h}{m}\right) = \log_p \frac{1-q^m}{1-q} = \frac{d}{dx} \log_p\left(\frac{1-q^m}{1-q}\right)^{x-1}$$

for any positive integer m (*not necessarily prime to* p),
(3)

$$\psi_{p,q}(x) - \psi_{p,1/q}(1-x) = -\log_p q = \frac{d}{dx} \log_p(-q)^{-x}.$$

PROOF. (1) This follows by Theorem 4 of [4]:

$$\psi_{p,q}(x+1) - \psi_{p,q}(x) = \lim_{N\to\infty} p^{-N}\left(\log_p \frac{1-q^{x+p^N}}{1-q} - \log_p \frac{1-q^x}{1-q}\right)$$
$$= \frac{d}{dx} \log_p \frac{1-q^x}{1-q}.$$

(2) By Theorem 1(ii) of [4], $\psi_{p,q}(x)$ can also be written (for any m) as

$$\lim_{N\to\infty} \frac{1}{mp^N} \sum_{0<j<mp^N} \log_p \frac{1-q^{x+j}}{1-q}.$$

Then the left side in (2) equals

$$\lim_{N\to\infty}\left(\frac{1}{mp^N} \sum_{0<j<mp^N} \log_p \frac{1-q^{x+j}}{1-q} \right.$$
$$\left. - \frac{1}{mp^N} \sum_{0<j<p^N} \sum_{0<h<m} \log_p \frac{1-q^{m((x+h)/m+j)}}{1-q^m}\right)$$
$$= \lim_{N\to\infty} \frac{1}{mp^N} \sum_{0<j<mp^N}\left(\log_p \frac{1-q^{x+j}}{1-q} - \log_p \frac{1-q^{x+j}}{1-q^m}\right) = \log_p \frac{1-q^m}{1-q}.$$

(3)

$$\psi_{p,q}(x) - \psi_{p,1/q}(1-x)$$
$$= \lim_{N\to\infty} p^{-N} \sum_{0<j<p^N}\left(\log_p \frac{1-q^{x+j}}{1-q} - \log_p \frac{1-q^{-(1-x+p^N-1-j)}}{1-q^{-1}}\right)$$
$$= -\log_p q + \lim_{N\to\infty} p^{-N} \sum_{0<j<p^N} \log_p \frac{1-q^{x+j}}{1-q^{x+j-p^N}}.$$

Since

$$\log_p \frac{1-q^{x+j}}{1-q^{x+j-p^N}} = -\log_p\left(1 - \frac{q^{x+j}}{1-q^{x+j}}\left(q^{-p^N}-1\right)\right)$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} \frac{q^{(x+j)k}}{\left(1-q^{x+j}\right)^k}\left(q^{-p^N}-1\right)^k,$$

and

$$q^{-p^N} - 1 = (1+t)^{-p^N} - 1 = -p^N t + p^N(p^N+1)t^2/2 - \cdots,$$

it quickly follows that the last limit is zero. Q.E.D.

Finally, we give q-extensions of Diamond's p-adic Euler constants [**4**]. If $r, f \in \mathbf{Z}, f \geqslant 1$, and if $\operatorname{ord}_p(r/f) < 0$, then define

$$\gamma_{p,q}(r,f) = -\lim_{N\to\infty} \frac{1}{fp^N} \sum_{\substack{0<j<fp^N \\ j\equiv r \pmod f}} \log_p \frac{1-q^j}{1-q}.$$

Also set

$$\gamma_{p,q} = \frac{p}{p-1}\sum_{j=1}^{p-1} \gamma_{p,q}(j,p) = -\frac{p}{p-1}\lim_{N\to\infty} p^{-N} \sum_{0<j<p^N}{}' \log_p \frac{1-q^j}{1-q}.$$

Then it is easy to prove the following two theorems, which generalize Theorem 14 in [**4**].

THEOREM 5. (1) *If* $d|(r,f)$, *then*

$$f\gamma_{p,q}(r,f) = (f/d)\gamma_{p,q^d}(r/d, f/d) - \log_p \frac{1-q^d}{1-q}.$$

(2) $\gamma_{p,q}(r,f) = \gamma_{p,1/q}(f-r,f) + (1/f)\log_p q$.

(3) *If b is a positive integer, then* $\gamma_{p,q}(r,f) = \sum_{j=0}^{b-1} \gamma_{p,q}(r+jf, bf)$.

THEOREM 6. (1) *If* $\operatorname{ord}_p(r/f) < 0$ *and* $0 < r < f$, *then*

$$\psi_{p,q^f}(r/f) = -\log_p \frac{1-q^f}{1-q} - f\gamma_{p,q}(r,f).$$

(2) $\psi^*_{p,q}(0) = \Gamma'_{p,q}(0) = -(1-1/p)\gamma_{p,q}$.

REMARKS. 1. In [**4**] Diamond denotes a limit of the form $\lim_{N\to\infty} p^{-N}\sum_{0<j<p^N} f(j,x)$ by $\int f(u,x)\,du$. This is the Riemann sum definition of the integral over $\mathbf{Z}_p$ with respect to the Haar distribution $\mu_{\text{Haar}}(j+p^N\mathbf{Z}_p) = p^{-N}$ if one chooses j as the representative "point" in the "interval" $j+p^N\mathbf{Z}_p$; since μ_{Haar} is not bounded, the limit of the Riemann sums does not exist independently of the choice of representative. The relationship between μ_{Haar} and the μ_z in [**9**] given by $\mu_z(j+p^N\mathbf{Z}_p) = z^j/(1-z^{p^N})$ is as follows. If we choose $z = q$ with $|q-1|_p < 1$ (but q not a pth-power root of 1), then μ_q is an unbounded distribution, and $d\mu_{\text{Haar}} = -q^{-u}\log_p q\, d\mu_q = (d/du)(q^{-u})\,d\mu_q$ in the sense that

$$\lim_{N\to\infty} \sum_{0<j<p^N} f(j, x)\mu_{\text{Haar}}(j + p^N\mathbf{Z}_p)$$
$$= \lim_{N\to\infty} \sum_{0<j<p^N} f(j, x)(-q^{-j}\log_p q)\mu_q(j + p^N\mathbf{Z}_p).$$

2. If we wanted to construct a q-extension $G_{p,q}$ of G_p having the form $G_{p,q}(x) = \lim_{N\to\infty} p^{-N}\Sigma_{0\leqslant j<p^N} f(x + j)$, we would have to find an f whose derivative is $\log_p((1 - q^x)/(1 - q))$. It is not clear what a natural choice for such an f might be. However, one could use the technique in [9] to define a "twisted" $G_{p,q}$ by

$$G_{p,q,\xi}(x) = -\int_{\mathbf{Z}_p} \log_p \frac{1 - q^{x+u}}{1 - q}\, d\mu_\xi(u) \quad \text{for } x \in D_q,$$

where $\xi^r = 1, \xi \neq 1, p\nmid r$, and $\mu_\xi(j + p^N\mathbf{Z}_p) = \xi^j/(1 - \xi^{p^N})$. This function satisfies

(1)
$$\xi G_{p,q,\xi}(x + 1) - G_{p,q,\xi}(x) = \log_p \frac{1 - q^x}{1 - q},$$

(2)
$$G_{p,q,\xi}(x) - \sum_{h=0}^{m-1} \xi^h G_{p,q^m,\xi^m}\left(\frac{x + h}{m}\right) = \frac{1}{1 - \xi} \log_p \frac{1 - q^m}{1 - q},$$

(3)
$$G_{p,q,\xi}(x) + \xi^{-1}G_{p,q^{-1},\xi^{-1}}(1 - x) = \frac{1}{1 - \xi} \log_p q.$$

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